Exploring Tessellations With Regular and Irregular Polygons

Mosaics are often made of repeating patterns of tiles. What patterns do you see in the design?

Many mosaic tile designs are made from shapes that cover the area, or the plane, without overlapping or leaving gaps. These patterns are called **tiling patterns** or **tessellations**. Covering the plane in this way is called **tiling the plane**.

Which shapes can you use to tile or tessellate the plane?

1. Copy the following table into your notebook.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Regular or Irregular Polygon?</th>
<th>Measure of Each Interior Angle</th>
<th>Prediction: Will the shape tile the plane?</th>
<th>Result: Does the shape tile the plane?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilateral triangle</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Isosceles triangle</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Square</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular pentagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular hexagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular octagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Irregular quadrilateral</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Irregular pentagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Irregular hexagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. a) Select an equilateral triangle block. Is this a regular or irregular polygon? Record your answer in the table.

b) Measure each interior angle and record your measurements in the table.

c) Predict whether the shape will tile the plane. Record your prediction in the table.

3. Trace the outline of the equilateral triangle. Move the triangle to a new position, so that the two triangles share a common side. Trace the outline of the triangle again. Continue to see if the shape tiles the plane. Record your conclusion in the table.

4. Use the same method to find out if the isosceles triangle, square, regular pentagon, regular hexagon, and regular octagon tile the plane. Record your results in the table.

5. Cut out the shape of an irregular quadrilateral.
   a) Predict whether the shape will tile the plane.
   b) Try to tile the plane with the shape. Record your results in the table.
   c) Repeat steps 5a) and 5b) using an irregular pentagon and an irregular hexagon of your own design.

Reflect on Your Findings

6. a) What regular shapes tile the plane? Explain why some regular shapes tile the plane but others do not. Hint: Look at the interior angle measures. Is there a pattern?

b) Explain why some irregular shapes tile the plane but others do not.
Example: Identify Shapes That Tessellate the Plane

Do these polygons tessellate the plane? Explain why or why not.

**Solution**

**a)** Arrange the squares along a common side. The rotated squares do not overlap or leave gaps when you try to form them into a tessellation. Shape A can be used to tessellate the plane.

Check:
Each of the interior angles where the vertices of the polygons meet is 90°. The sum of the four angles is $90° + 90° + 90° + 90° = 360°$. This is equal to a full turn. The shape can be used to tessellate the plane.

**b)** Arrange the pentagons along a common side. The irregular pentagons overlap or leave gaps when you try to form them into a tessellation. Shape B cannot be used to tessellate the plane.

Check:
Each of the interior angles where the vertices of the polygons meet is 96°. The sum of the four angles is $96° + 96° + 96° + 96° = 384°$. This is more than a full turn. The shape cannot be used to tessellate the plane.
A tiling pattern or tessellation is a pattern that covers a plane without overlapping or leaving gaps.

- Only three types of regular polygons tessellate the plane.
- Some types of irregular polygons tessellate the plane.
- Regular and irregular polygons tessellate the plane when the interior angle measures total exactly 360° at the point where the vertices of the polygons meet.

1. Draw three types of regular polygons that tessellate the plane. Justify your choices.

2. What are two types of irregular polygons that can be used to tessellate the plane? Explain your choices to a friend.

3. Megan is tiling her kitchen floor. Should she choose ceramic tiles in the shape of a regular octagon? Explain how you know.

Which of the following shapes can be used to tessellate the plane? Explain your reasoning.

- a) 
  - 105° 125° 130° 120° 120° 120°
- b) 
  - 120° 60° 120° 60°
- c) 
  - 50° 70° 60° 60° 70°

- 105° + 75° + 75° + 105° = 360°
Check Your Understanding

Practise

For help with #4 to #7, refer to the Example on page 448.

4. Do these regular polygons tessellate the plane? Explain why or why not.
   a) 
   b) 

5. Use this shape to tessellate the plane. Show and colour the result on grid paper.

6. Tessellate the plane with an isosceles triangle. Use colours or shading to create an interesting design on grid paper.

7. Describe three tessellating patterns that you see at home or at school. What shapes make up the tessellation?

Apply

8. Jared is painting a mosaic on one wall of her bedroom that is made up of tessellating equilateral triangles. Describe two different tessellation patterns that Jared could use. Use triangular dot paper to help you describe the tessellations.

9. Patios are often made from interlocking rectangular bricks. The pattern shown below is called herringbone.

   ![Herringbone](image)

   On grid paper, create two different patio designs from congruent rectangular bricks.

10. Some pentagons can be used to tessellate the plane.
   a) Describe a pentagon that will tessellate the plane. Explain how it tessellates the plane.
   b) Compare your pentagon with those of your classmates. How many different tessellating pentagons did you and your classmates find?

11. A pentomino is a shape made up of five squares. Choose two of the following pentominoes and try to make a tessellation with each one. Do each of your pentominoes make a tessellation? Explain why or why not.

   ![Pentomino](image)
12. Sarah is designing a pattern for the hood and cuffs of her new parka. She wants to use a regular polygon in the design and three different colours. Use grid paper to create two different designs that Sarah might use. Colour your designs.

13. The diagram shows a tessellation of squares. A dot has been added to the centre of each square. The dots are joined by dashed segments perpendicular to common sides. The result is another tessellation, which is called the dual of the original tessellation.

14. Identify two different regular polygons that can be used together to create a tessellating pattern. Draw a tessellation on grid paper using the two polygons.

Extend

Did You Know?
Many Islamic artists make very intricate geometric decorations and are experts at tessellation art.

MATH LINK
This tiling pattern is from Alhambra, a Moorish palace built in Granada, Spain. Four different tile shapes are used to create this pattern.

a) Describe the four shapes. Are they regular or irregular polygons?

b) Use templates to trace the shapes onto cardboard or construction paper.

c) Cut out ten of each shape and use some or all of them to create at least two different tile mosaics. Use each of the four shapes in your mosaics.

WWW Web Link
To generate tessellations on the computer, go to www.mathlinks8.ca and follow the links.
In section 12.1 you created simple tessellating patterns using regular and irregular polygons. Tessellations can also be made by combining regular or irregular polygons and then transforming them. Do you recognize the polygons used in this tessellation? What transformations were used to create the pattern?

**Explore the Math**

**How can you create a tessellation using transformations?**

1. Draw a regular hexagon on a piece of paper using a pattern block or cardboard cutout. Cut out the hexagon and glue it to a sheet of cardboard or construction paper.

2. Draw two equilateral triangles on a piece of paper using a pattern block or cardboard cutout. Make sure that the side lengths of the triangles are the same as the side lengths of the hexagon. Cut out the triangles and glue them to a sheet of cardboard or construction paper so that they are attached to the sides of the hexagon as shown.

3. Cut out the combined shape. Trace the shape on a new sheet of paper.
4. Translate the shape so that the hexagon fits into the space formed by the two triangles. Trace around the translated shape and repeat two more times. What other ways can you translate the shape?

5. Translate the combined piece vertically and horizontally so that the base of the hexagon is now at the top of one of the triangles.

Reflect on Your Findings

6. a) Describe how to use translations to create tessellations.
    
    b) What other transformations could you use to get the same pattern as in #5? Explain the difference.

Example: Identify the Transformation

a) What polygons and what transformations are used to create this tessellation?

b) Does the area of the tessellating tile change during the tessellation?
Solution

a) The tessellation is made from a tessellating tile consisting of a hexagon with two squares and two equilateral triangles. The tessellating tile is then translated vertically and horizontally. This tessellation is created using translations.

b) The area of the tessellating tile remains the same throughout the tessellation. There are no gaps or overlapping pieces.

Show You Know

What transformation was used to create this tessellation? Explain your reasoning.

Key Ideas

- Tessellations can be made with two or more polygons as long as the interior angles where the vertices of the polygons meet total exactly 360°.
- Two types of transformations commonly used to create tessellations are
  - translations
  - reflections
- The area of the tessellating tile remains the same after it has been transformed to create a tessellation.

Communicate the Ideas

1. Brian missed today’s class. How would you explain to him why some tessellating patterns made using translations could also be made using reflections?
2. Ashley and Vijay are trying to figure out how this tessellation was made. Whose answer is correct? Explain.

Ashley says: The tessellation is based on reflecting the blue triangles across the red dodecagon.

Vijay says: The tessellation is based on translating the red dodecagon with 2 blue triangles.

3. Identify the two regular polygons used to create each tessellation.

a)

b)

c)

4. What type of transformation could be used to create each tessellation in #3?

5. The diagram shows a garden path made from irregular 12-sided bricks.

a) Explain why the 12-sided brick tessellates the plane.

b) Use grid paper to design an irregular ten-sided brick that could be used to make a path.

c) Explain why your ten-sided brick tessellates the plane.

d) Use grid paper to design an irregular six-sided brick that could be used to tessellate the plane.

e) Explain why your six-sided brick tessellates the plane.
6. Simon is designing a wallpaper pattern that tessellates. He chooses to use the letter “T” as the basis of his pattern. Create two tessellations using the three coloured letters shown.

7. Priya is designing a kitchen tile that uses two different regular polygons. She then uses two different translations to create a tessellation. Use grid paper to design a tile that Priya could use. Show how it tiles the plane.

8. Barbara wants to make a quilt using the two polygons shown. Will she be able to create a tessellating pattern using these shapes? Explain.

9. An equilateral triangle is called a reptile (an abbreviation for “repeating tile”) because four equilateral triangles can be arranged to form a larger equilateral triangle.

Which of these figures are reptiles? Use grid paper to draw the larger figure for each reptile.

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**Math Link**

Many quilt designs are made using tessellating shapes. This quilt uses fabric cut into triangles that are sewn together to form squares. The squares are then translated vertically and horizontally.

Design your own quilt square using one or more regular tessellating polygons. Create an interesting design based on patterns or colours.
Constructing Tessellations Using Rotations

Focus on...

After this lesson, you will be able to...

- identify how rotations can be used to create a tessellation
- create tessellating patterns using two or more polygons

Did You Know?

Professor Ronald Resch of the University of Utah built the world’s largest pysanka from 3500 pieces of aluminum. It is located in Vegreville, Alberta; weighs 2300 kg; is 9.4 m high, 7 m long, and 5.5 m wide; and turns in the wind like a weather vane!

Explore the Math

How can you create tessellations using rotations?

1. Draw an equilateral triangle with side lengths of 4 to 5 cm on a piece of paper. Cut out the triangle and glue it to a sheet of cardboard or construction paper to create a tile.

2. Trace around your tile on a piece of paper.
3. Rotate the tile 60° about one vertex until the edge of the tile falls along the edge of the previous tracing as shown. Trace around the tile again.

4. Repeat #3 until a full turn has been made.
   a) What shape did you create?
   b) How many times did you have to rotate the tile to create this shape?

5. Add colour and designs to the tessellation to make a piece of art.

6. How could you continue to use rotations to make a larger tessellation?

Reflect on Your Findings

7. a) Describe how to use rotating polygons to create tessellations.
   b) What types of polygons can be used to make tessellations based on rotations? Explain.

Example: Identify the Transformation
What polygons and what transformation could be used to create this tessellation?

Solution
The tessellating tile is made up of a regular hexagon that has been rotated three times to make a complete turn. The three hexagons forming this tile can be translated horizontally and diagonally to enlarge the tessellation.

Show You Know
What polygons and transformations could be used to create this tessellation? Explain how you know.
Key Ideas

- Tessellations can be made with two or more polygons as long as the interior angles where the polygons meet total exactly 360°.
- Rotations can be used to create tessellations.

Communicate the Ideas

1. When creating a tessellation using rotations, why is it important for the sum of the angle measures at the point of rotation to equal 360°? Explain.

2. Describe to a partner how to use rotating polygons to create tessellations.

Check Your Understanding

Practise

For help with #3 and #4, refer to the Example on page 458.

3. Identify the polygons used to create each tessellating tile.
   a) 
   b) 
   c) 

4. What transformations could be used to create each tessellation in #3?

Apply

5. Examine the piece of stained glass.
   a) Describe the transformation(s) used to make this pattern.
   b) If you were using this pattern to tile the plane, what modifications would you have to make?

6. Design your own stained-glass window on grid paper. Describe the steps you followed to create the pattern.

7. Create a tessellation using two different regular polygons and rotations.
8. Which of the following shapes tessellate? Explain how you know a shape will or will not tessellate.

9. The diagram shows one arrangement of three or more polygons that can be used to create tessellations using rotations. One triangle and two dodecagons can be used because the angles at each vertex total 360° where they join. This is represented as (3, 12, 12). The table shows the features of this tessellation, for Shape 1.

<table>
<thead>
<tr>
<th>Tessellations Involving Three Regular Polygons</th>
<th>Shape 1</th>
<th>Shape 2</th>
<th>Shape 3</th>
<th>Shape 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle (60°)</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Square (90°)</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pentagon (108°)</td>
<td></td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hexagon (120°)</td>
<td></td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Octagon (135°)</td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Dodecagon (150°)</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Number of Sides</td>
<td>(3, 12, 12)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sum of Angles</td>
<td>60 + 2(150)</td>
<td></td>
<td>360°</td>
<td></td>
</tr>
</tbody>
</table>

a) Copy the table into your notebook. Complete the table for Shape 2 for the diagram shown.

b) Complete the table for Shapes 3 and 4, using different combinations of three or more regular polygons that total 360°.

c) Create construction paper or cardboard cutouts of the regular polygons from part b). Try to tessellate the plane using the combinations that you believe will work.

MATH LINK

Create your own pysanka design based on tessellating one or more polygons. Use at least one rotation in your design. Trace your design on grid paper, and colour it. Make sure it is the correct size to fit on an egg. If you have time, decorate an egg with your pysanka design.

Web Link

To see examples of pysankas, go to www.mathlinks8.ca and follow the links.
Focus on...  
After this lesson, you will be able to...
- create tessellations from combinations of regular and irregular polygons
- describe the tessellations in terms of the transformation used to create them

Explore the Math

In the previous sections, you created tessellating patterns using regular and irregular polygons. When Escher created his tessellations, he did so in a variety of ways. Look at the two Escher works. What is different about the tessellations?

Creating Escher-Style Tessellations

1. Draw an equilateral triangle with 6-cm sides on a blank piece of paper. Cut out the triangle and glue it to a sheet of cardboard or construction paper. Cut out the triangle again.

2. Inside the triangle, draw a curve that connects two adjacent vertices. Cut along the curve to remove a piece from one side of the triangle.

3. Rotate the piece you removed 60° counterclockwise about the vertex at the top end of the curve. This rotation moves the piece to another side of the triangle. Tape the piece in place to complete your tile.
4. To tessellate the plane, draw around the tile on a piece of paper. Then, rotate and draw around the tile over and over until you have a design you like.

5. Add colour and designs to the tessellation to make a piece of art.

6. Repeat steps 1 through 5 using a parallelogram and translations to create another Escher-style drawing.

Reflect on Your Findings

7. You can use transformations to create Escher-style tessellations just as you did with regular and irregular polygons.
   a) Describe how to use rotations to create Escher-style tessellations.
   b) What do you notice about the sum of the angle measures at the vertices where the tessellating tiles meet?
   c) How does the area of the modified tile compare with the area of the original polygon? Explain.

Example: Identify the Transformation Used in a Tessellation

What transformation was used to create each of the following tessellations?
Solution

Tessellation A is made up of triangles that have been rotated to form a hexagon. This tessellation is made using rotations.

Tessellation B is made up of figures that alternate gold to black and then repeat horizontally across the drawing. This tessellation is made using translations.

Show You Know

What transformation was used to create this tessellation? Explain your answer.

Key Ideas

- You can create Escher-style tessellations using the same methods you used to create tessellations from regular or irregular polygons:
  - Start with a regular or irregular polygon.
  - The area of the tessellating tile must remain unchanged—any portion of the tile that is cut out must be reattached to the tile so that it fits with the next tile of the same shape.
  - Make sure there are no overlaps or gaps in the pattern.
  - Make sure interior angles at vertices total exactly 360°.
  - Use transformations to tessellate the plane.
**Communicate the Ideas**

1. When creating a tile for an Escher-style tessellation, the original polygon is cut up. How do you know the area of the original polygon is maintained?

2. Rico believes that he can use this tile to create an Escher-style tessellation. Is he correct? Explain.

3. Tessellations must have no gaps or overlaps. What other two properties must be maintained when creating Escher-style tessellations?

**Check Your Understanding**

**Practise**

For help with #4 to #7, refer to the Example on pages 462–463.

4. Identify the transformations used to create each tessellation.

a) ![Tessellation A]

b) ![Tessellation B]

5. Identify the original shape from which each tile was made for each tessellation in #4.
6. Identify the transformations used to create each tessellation.

   a) 

   b) 

7. Identify the original shape from which each tile was made for each tessellation in #6.


11. Escher also used impossible figures in his art, as shown.

   a) What impossible figures were used in the drawing?

   b) Research other examples of Escher’s art that include impossible figures.

**MATH LINK**

Use an Escher-style tessellation to create a design for a binder cover, wrapping paper, a border for writing paper, or a placemat.

**WWW Web Link**

To see examples of Escher’s art, go to www.mathlinks8.ca and follow the links.